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Glafka-2004: Non-Commutative Worlds

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This paper summarizes and gives new extensions of previous work of the author and collaborative work of the author with Pierre Noyes. In the present paper we give a new generalization of the Feynman-Dyson derivation of electromagnetism in a non-commutative context. In this form, the theory extends to gauge fields and is entirely a consequence of a choice of the definition of derivatives as commutators and the choice of relationship between temporal and spatial derivatives. The paper uses diagrammatic techniques and discusses these issues in the context of discrete physical models.

KEY WORDS: discrete derivative; commutator; Lie algebra; Hamilton's equations; curvature; gauge field; electromagnetism; metric; Jacobi identity.

1. INTRODUCTION TO NON-COMMUTATIVE WORLDS

Aspects of gauge theory, Hamiltonian mechanics and quantum mechanics arise naturally in the mathematics of a non-commutative framework for calculus and differential geometry. This paper consists in four sections including the introduction. The introduction sketches our general results in this domain. The second section gives a derivation of a generalization of the Feynman-Dyson derivation of electromagnetism using our non-commutative context and using diagrammatic techniques. The introduction is based on the paper Kauffman (2004). The second section is a new approach to issues in Kauffman (2004). The third section discusses relationships with differential geometry. The fourth section discusses, in more depth, relationships with gauge theory and differential geometry.

Constructions are performed in a Lie algebra $\mathcal{A}$. One may take $\mathcal{A}$ to be a specific matrix Lie algebra, or abstract Lie algebra. If $\mathcal{A}$ is taken to be an abstract Lie algebra, then it is convenient to use the universal enveloping algebra so that the Lie product can be expressed as a commutator. In making general constructions

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of operators satisfying certain relations, it is understood that one can always begin with a free algebra and make a quotient algebra where the relations are satisfied.

On $\mathcal{A}$, a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed N in $\mathcal{A}$ one defines

$$\nabla_N : \mathcal{A} \longrightarrow \mathcal{A}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$

∇_N is a derivation satisfying the Leibniz rule.

$$\nabla_N(FG) = \nabla_N(F)G + F\nabla_N(G).$$

There are many motivations for replacing derivatives by commutators. If $f(x)$ denotes (say) a function of a real variable x, and $\tilde{f}(x) = f(x+h)$ for a fixed increment h, define the *discrete derivative* Df by the formula $Df = (\tilde{f} - f)/h$, and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator J with the property that

$$fJ = J\tilde{f}.$$

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$

It follows at once that

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g).$$

Note that

$$\nabla(f) = \frac{(J\tilde{f} - Jf)}{h} = \frac{(fJ - Jf)}{h} = \left[f, \frac{J}{h}\right].$$

In the extended algebra, discrete derivatives are represented by commutators, and satisfy the Leibniz rule. One can regard discrete calculus as a subset of non-commutative calculus based on commutators.

In $\mathcal{A}$ there are as many derivations as there are elements of the algebra, and these derivations behave quite wildly with respect to one another. If one takes the concept of *curvature* as the non-commutation of derivations, then $\mathcal{A}$ is a highly curved world indeed. Within $\mathcal{A}$ one can build a tame world of derivations that mimics the behaviour of flat coordinates in Euclidean space. The description of the structure of $\mathcal{A}$ with respect to these flat coordinates contains many of the equations and patterns of mathematical physics.

The flat coordinates X_i satisfy the equations below with the P_j chosen to represent differentiation with respect to X_j.:

$$[X_i, X_j] = 0$$

$$[P_i, P_j] = 0$$

$$[X_i, P_j] = \delta_{ij}.$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F/\partial X_i = [F, P_i],$$

$$\hat{\partial}_i F = \partial F/\partial P_i = [X_i, F].$$

Temporal derivative is represented by commutation with a special (Hamiltonian) element H of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take $i\hbar dA/dt = [A, H]$.) These non-commutative coordinates are the simplest flat set of coordinates for description of temporal phenomena in a non-commutative world. Note that *Hamilton's equations are a consequence of these definitions*. The very short proof of this fact is given below.

1.1. Hamilton's Equations

$$\frac{dP_i}{dt} = [P_i, H] = -[H, P_i] = -\frac{\partial H}{\partial X_i}$$

$$\frac{dX_i}{dt} = [X_i, H] = \frac{\partial H}{\partial P_i}.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

1.1.1. Discrete Measurement

Consider a time series $\{X, X', X'', \ldots\}$ with commuting scalar values. Let

$$\dot{X} = \nabla X = JDX = \frac{J(X' - X)}{\tau}$$

where τ is an elementary time step (If X denotes a times series value at time t, then X' denotes the value of the series at time $t + \tau$.). The shift operator J is defined by the equation $XJ = JX'$ where this refers to any point in the time series so that $X^{(n)}J = JX^{(n+1)}$ for any non-negative integer n. Moving J across a variable from right to left, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.

This derivative ∇ also fits a significant pattern of discrete observation. Consider the act of observing X at a given time and the act of observing (or obtaining) DX at a given time. Since X and X' are ingredients in computing $(X' - X)/\tau$, the numerical value associated with DX, it is necessary to let the clock tick once, Thus, if one first observes X and then obtains DX, the result is different (for the X measurement) if one first obtains DX, and then observes X. In the second case, one finds the value X' instead of the value X, due to the tick of the clock.

1. Let $\dot{X}X$ denote the sequence: observe X, then obtain $\dot{X}$.
2. Let $X\dot{X}$ denote the sequence: obtain $\dot{X}$, then observe X.

The commutator $[X, \dot{X}]$ expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[X, \dot{X}] = X\dot{X} - \dot{X}X = \frac{J(X' - X)^2}{\tau}.$$

Thus one can interpret the equation

$$[X, \dot{X}] = Jk$$

(k a constant scalar) as

$$\frac{(X' - X)^2}{\tau} = k.$$

This means that the process is a walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$

where k is a constant. In other words, one has the equation

$$k = \frac{\Delta^2}{\tau}.$$

This is the diffusion constant for a Brownian walk. A walk with spatial step size Δ and time step τ will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This shows that the diffusion constant of a Brownian process is a structural property of that process, independent of considerations of probability and continuum limits.

1.1.2. Heisenberg/Schroedinger Equation

Here is how the Heisenberg form of Schroedinger's equation fits in this context. Let $J = (1 + H\Delta t/i\hbar)$. Then $\nabla\psi = [\psi, J/\Delta t]$, and we calculate

$$\nabla\psi = \psi\left[\frac{(1 + H\Delta t/i\hbar)}{\Delta t}\right] - \left[\frac{(1 + H\Delta t/i\hbar)}{\Delta t}\right]\psi = \frac{[\psi, H]}{i\,\hbar}.$$

This is exactly the form of the Heisenberg equation.

1.1.3. Dynamics and Gauge Theory

One can take the general dynamical equation in the form

$$\frac{dX_i}{dt} = \mathcal{G}_i$$

where $\{\mathcal{G}_1, \ldots, \mathcal{G}_d\}$ is a collection of elements of $\mathcal{A}$. Write $\mathcal{G}_i$ relative to the flat coordinates via $\mathcal{G}_i = P_i - A_i$. This is a definition of A_i and $\partial F/\partial X_i = [F, P_i]$. The formalism of gauge theory appears naturally. In particular, if

$$\nabla_i(F) = [F, \mathcal{G}_i],$$

then one has the curvature

$$[\nabla_i, \nabla_j]F = [R_{ij}, F]$$

and

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

This is the well-known formula for the curvature of a gauge connection. Aspects of geometry arise naturally in this context, including the Levi-Civita connection (which is seen as a consequence of the Jacobi identity in an appropriate non-commutative world).

With $\dot{X}_i = P_i - A_i$, the commutator $[X_i, \dot{X}_j]$ takes the form

$$[X_i, \dot{X}_j] = [X_i, P_j - A_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - [X_i, A_j] = g_{ij}.$$

Thus we see that the "gauge field" A_j provides the deviation from the Kronecker delta in this commutator. We have $[\dot{X}_i, \dot{X}_j] = R_{ij}$, so that these commutators represent the curvature.

One can consider the consequences of the commutator $[X_i, \dot{X}_j] = g_{ij}$, deriving that

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t,$$

where G_r is the analogue of a scalar field, F_{rs} is the analogue of a gauge field and Γ_{rst} is the Levi-Civita connection associated with g_{ij}. This decompositon of the acceleration is uniquely determined by the given framework. We shall give this derivation in Section 4.

In regard to thinking about the commutator $[X_i, \dot{X}_j] = g_{ij}$, It is worth noting that this equation is a consequence of the right choice of Hamiltonian. By this I mean, that in a given non-commutative world we choose an H in the algebra to represent the total (or discrete) time derivative so that $\dot{F} = [F, H]$ for any F. Suppose we have elements g_{ij} such that

$$[g_{ij}, X_k] = 0$$

and

$$g_{ij} = g_{ji}.$$

We choose

$$H = \frac{(g_{ij}P_iP_j + P_iP_jg_{ij})}{4}.$$

This is the non-commutative analog of the classical $H = (1/2)g_{ij}P_iP_j$. In Section 3, we show that this choice of Hamiltonian implies that $[X_i, \dot{X}_j] = g_{ij}$.

1.1.4. Feynman—Dyson Derivation

One can use this context to revisit the Feynman-Dyson derivation of electromagnetism from commutator equations, showing that most of the derivation is independent of any choice of commutators, but highly dependent upon the choice of definitions of the derivatives involved. Without any assumptions about initial commutator equations, but taking the right (in some sense simplest) definitions of the derivatives one obtains a significant generalization of the result of Feynman-Dyson.

1.1.5. Electromagnetic Theorem

See Section 2. With the appropriate [see below] definitions of the operators, and taking

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad B = \dot{X} \times \dot{X} \text{ and } E = \partial_t\dot{X}, \quad \text{one has}$$

1. $\ddot{X} = E + \dot{X} \times B$
2. $\nabla \bullet B = 0$

3. $\partial_t B + \nabla \times E = B \times B$
4. $\partial_t E - \nabla \times B = \left(\partial_t^2 - \nabla^2\right)\dot{X}$

The key to the proof of this Theorem is the definition of the time derivative. This definition is as follows

$$\partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F) = \dot{F} - \Sigma_i \dot{X}_i [F, \dot{X}_i]$$

for all elements or vectors of elements F. The definition creates a distinction between space and time in the non-commutative world. In the non-commutative world, we are give the process derivative $\dot{F} = [F, H]$, conceived originally as a discrete difference ratio. The elements of the non-commutative world are subject to this temporal variation, but they are not functions of a "time variable" t. The concept of a time variable is a classical notion that we bring partially into the non-commutative context by defining the notion of a partial derivative $\partial_t F$.

A calculation (done diagrammatically in Fig. 3) reveals that

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

This suggests taking $E = \partial_t \dot{X}$ as the electric field, and $B = \dot{X} \times \dot{X}$ as the magnetic field so that the Lorentz force law

$$\ddot{X} = E + \dot{X} \times B$$

is satisfied.

This result is applied to produce many discrete models of the Theorem. These models show that, just as the commutator $[X, \dot{X}] = Jk$ describes Brownian motion (constant step size processes) in one dimension, a generalization of electromagnetism describes the interaction of triples of time series in three dimensions.

Remark 1. While there is a large literature on non-commutative geometry, emanating from the idea of replacing a space by its ring of functions, work discussed herein is not written in that tradition. Non-commutative geometry does occur here, in the sense of geometry occuring in the context of non-commutative algebra. Derivations are represented by commutators. There are relationships between the present work and the traditional non-commutative geometry, but that is a subject for further exploration. In no way is this paper intended to be an introduction to that subject. The present summary is based on (Kauffman, 1991; Kauffman and Noyes, 1996a,b; Kauffman *et al.*, in preparation; Kauffman, 1996, 1998a,b, 1999, 2003, 2004) and the references cited therein.

The following references in relation to non-commutative calculus are useful in comparing with the present approach (Connes, 1990; Dimakis and Müller-Hoissen, 1992; Forgy, 2002; Müller-Hoissen, 1998). Much of the present work is the fruit of a long series of discussions with Pierre Noyes, influenced at critical points by Tom Etter and Keith Bowden. The paper Montesinos and Perez-Lorenzana

(1999) also works with minimal coupling for the Feynman-Dyson derivation. The first remark about the minimal coupling occurs in the original paper by Dyson (1990), in the context of Poisson brackets. The paper Hughes (1992) is worth reading as a companion to Dyson. It is the purpose of this summary to indicate how non-commutative calculus can be used in foundations.

2 GENERALIZED FEYNMAN DYSON DERIVATION

In this section we assume that specific time-varying coordinate elements X_1, X_2, X_3 of the algebra $\mathcal{A}$ are given. *We do not assume any commutation relations about* X_1, X_2, X_3.

In this section we no longer avail ourselves of the commutation relations that are in back of the original Feynman-Dyson derivation. We do take the definitions of the derivations from that previous context. Surprisingly, the result is very similar to the one of Feynman and Dyson, as we shall see.

Here $A \times B$ is the non-commutative vector cross product:

$$(A \times B)_k = \Sigma_{i,j=1}^{3} \epsilon_{ijk} A_i B_j.$$

(We will drop this summation sign for vector cross products from now on.) Then, with $B = \dot{X} \times \dot{X}$, we have

$$B_k = \epsilon_{ijk} \dot{X}_i \dot{X}_j = \left(\frac{1}{2}\right) \epsilon_{ijk} [\dot{X}_i, \dot{X}_j].$$

The epsilon tensor ϵ_{ijk} is defined for the indices $\{i, j, k\}$ ranging from 1 to 3, and is equal to 0 if there is a repeated index and is ortherwise equal to the sign of the permutation of 123 given by ijk. We represent dot products and cross products in diagrammatic tensor notation as indicated in Figs. 1 and 2. In Fig. 1 we indicate the epsilon tensor by a trivalent vertex. The indices of the tensor correspond to labels for the three edges that impinge on the vertex. The diagram is drawn in the plane, and is well-defined since the epsilon tensor is invariant under cyclic permutation of its indices.

We will define the fields E and B by the equations

$$B = \dot{X} \times \dot{X} \text{ and } E = \partial_t \dot{X}.$$

We will see that E and B obey a generalization of the Maxwell Equations, and that this generalization describes specific discrete models. The reader should note that this means that a significant part of the *form* of electromagnetism is the consequence of choosing three coordinates of space, and the definitions of spatial and temporal derivatives with respect to them. The background process that is being described is otherwise aribitrary, and yet appears to obey physical laws once these choices are made.

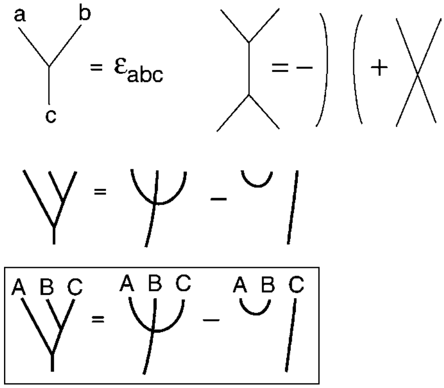


Fig. 1. Epsilon identity.

In this section we will use diagrammatic matrix methods to carry out the mathematics. In general, in a diagram for matrix or tensor composition, we sum over all indices labeling any edge in the diagram that has no free ends. Thus matrix multiplication corresponds to the connecting of edges between diagrams, and to the summation over common indices. With this interpretation of compositions, view the first identity in Fig. 1. This is a fundmental identity about the epsilon, and corresponds to the following lemma.

Lemma 1. *(View Fig. 1) Let* ϵ_{ijk} *be the epsilon tensor taking values* 0, 1 *and* -1 *as follows: When* ijk *is a permuation of* 123*, then* ϵ_{ijk} *is equal to the sign of*

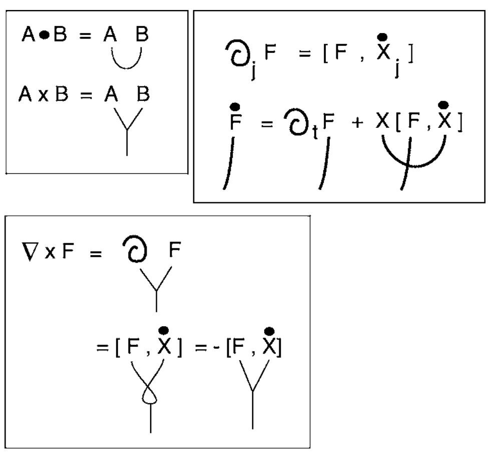
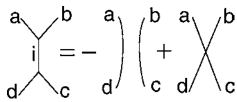


Fig. 2. Defining Derivatives.

the permutation. When ijk contains a repetition from $\{1, 2, 3\}$, then the value of epsilon is zero. Then ϵ satisfies the following identity in terms of the Kronecker delta.



$$\Sigma_i \epsilon_{abi}\epsilon_{cdi} = -\delta_{ad}\delta_{bc} + \delta_{ac}\delta_{bd}.$$

The proof of this identity is left to the reader. The identity itself will be referred to as the *epsilon identity*. The epsilon identity is a key structure in the work of this section, and indeed in all formulas involving the vector cross product.

The reader should compare the formula in this Lemma with the diagrams in Fig. 1. The first two diagram are two versions of the Lemma. In the third diagram the labels are capitalized and refer to vectors A, B and C. We then see that the epsilon identity becomes the formula

$$A \times (B \times C) = (A \bullet C)B - (A \bullet B)C$$

for vectors in three-dimensional space (with commuting coordinates, and a generalization of this identity to our non-commutative context. Refer to Fig. 2 for the diagrammatic definitions of dot and cross product of vectors. We take these definitions (with implicit order of multiplication) in the non-commutative context.

2.1. Remarks on the Derivatives

1. Since we do not assume that $[X_i, \dot{X}_j] = \delta_{ij}$, nor do we assume $[X_i, X_j] = 0$, it will not follow that E and B commute with the X_i.
2. We define

   $$\partial_i(F) = [F, \dot{X}_i],$$

   and the reader should note that, these spatial derivations are no longer flat in the sense of section 1 (nor were they in the original Feynman-Dyson derivation). See Fig. 2 for the diagrammatic version of this definition.
3. We define $\partial_t = \partial/\partial t$ by the equation

   $$\partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F) = \dot{F} - \Sigma_i \dot{X}_i [F, \dot{X}_i]$$

   for all elements or vectors of elements F. We take this equation as the global definition of the temporal partial derivative, even for elements that are not commuting with the X_i. This notion of temporal partial derivative ∂_t is a least relation that we can write to describe the temporal relationship of an arbitrary non-commutative vector F and the non-commutative coordinate vector X. See Fig. 2 for the diagrammatic version of this definition.

4. In defining

$$\partial_t F = \dot{F} - \Sigma_i \dot{X}_i \partial_i(F),$$

we are using the definition itself to obtain a notion of the variation of F with respect to time. The definition itself creates a distinction between space and time in the non-commutative world.

5. The reader will have no difficulty verifying the following formula:

$$\partial_t(FG) = \partial_t(F)G + F\partial_t(G) + \Sigma_i \partial_i(F)\partial_i(G).$$

This formula shows that ∂_t does not satisfy the Leibniz rule in our non-commutative context. This is true for the original Feynman-Dyson context, and for our generalization of it. All derivations in this theory that are defined directly as commutators do satisfy the Leibniz rule. Thus ∂_t is an operator in our theory that does not have a representation as a commutator.

6. We define divergence and curl by the equations

$$\nabla \bullet B = \Sigma_{i=1}^{3} \partial_i(B_i)$$

and

$$(\nabla \times E)_k = \epsilon_{ijk} \partial_i(E_j).$$

See Figs. 2 and 4 for the diagrammatic versions of curl and divergence.

Now view Fig. 3. We see from this Figure that it follows directly from the definition of the time derivatives (as discussed above) that

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

This is our motivation for defining

$$E = \partial_t \dot{X}$$

and

$$B = \dot{X} \times \dot{X}.$$

With these definitions in place we have

$$\ddot{X} = E + \dot{X} \times B,$$

giving an analog of the Lorentz force law for this theory.

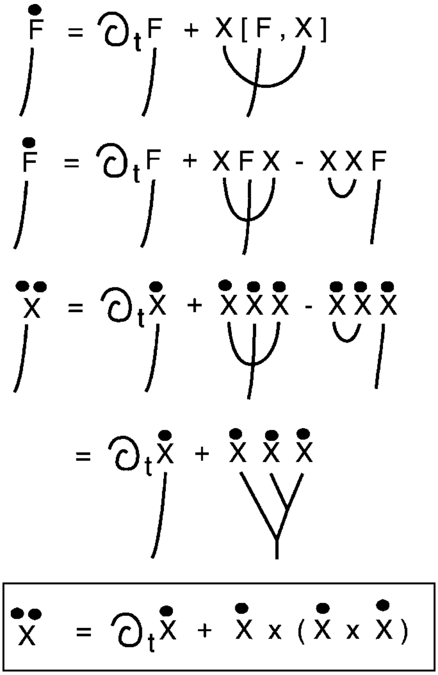


Fig 3. The formula for acceleration.

Just for the record, look at the following algebraic calculation for this derivative:

$$
\begin{aligned}
\dot{F} &= \partial_t F + \Sigma_i \dot{X}_i [F, \dot{X}_i] \\
&= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i \dot{X}_i F) \\
&= \partial_t F + \Sigma_i (\dot{X}_i F \dot{X}_i - \dot{X}_i F_i \dot{X}) + \dot{X}_i F_i \dot{X} - \dot{X}_i \dot{X}_i F
\end{aligned}
$$

Hence

$$\dot{F} = \partial_t F + \dot{X} \times F + (\dot{X} \bullet F)\dot{X} - (\dot{X} \bullet \dot{X})F$$

(using the epsilon identity). Thus we have

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}) + (\dot{X} \bullet \dot{X})\dot{X} - (\dot{X} \bullet \dot{X})\dot{X},$$

whence

$$\ddot{X} = \partial_t \dot{X} + \dot{X} \times (\dot{X} \times \dot{X}).$$

In Fig. 4, we give the derivation that B has zero divergence.

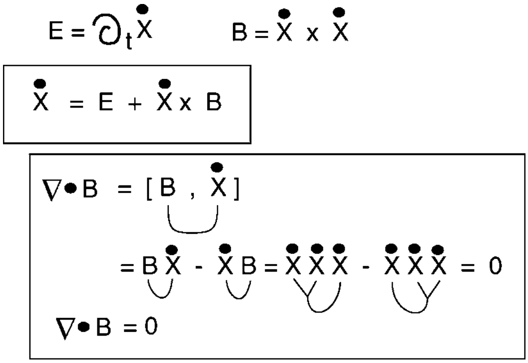


Fig 4. Divergence of B.

Figures 5 and 6 compute derivatives of B and the Curl of E, culminating in the formula

$$\partial_t B + \nabla \times E = B \times B.$$

In classical electromagnetism, there is no term $B \times B$. This term is an artifact of our non-commutative context. In discrete models, as we shall see at the end of this section, there is no escaping the effects of this term.

Finally, Fig. 7 gives the diagrammatic proof that

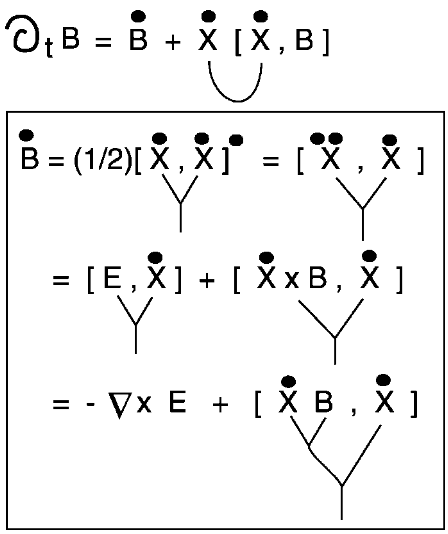


Fig 5. Computing $\dot{B}$.

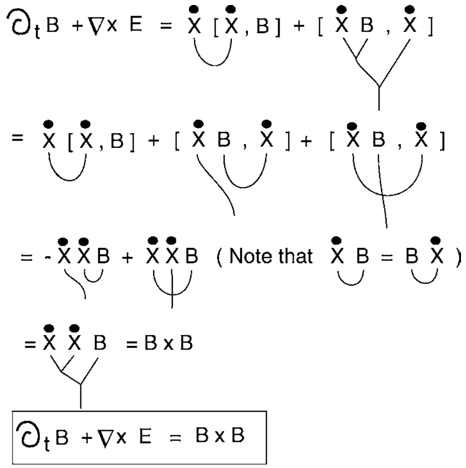


Fig 6. Curl of E.

$$\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2)\dot{X}.$$

This completes the proof of the Theorem below.

Electromagnetic Theorem. With the above definitions of the operators, and taking

$$\nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2, \quad B = \dot{X} \times \dot{X} \text{ and } E = \partial_t \dot{X} \text{ we have}$$

1. $\ddot{X} = E + \dot{X} \times B$
2. $\nabla \bullet B = 0$
3. $\partial_t B + \nabla \times E = B \times B$
4. $\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2)\dot{X}$

Remark 2. Note that this Theorem is a non-trivial generalization of the Feynman-Dyson derivation of electromagnetic equations. In the Feynman-Dyson case, one assumes that the commutation relations

$$[X_i, X_j] = 0$$

and

$$[X_i, \dot{X}_j] = \delta_{ij}$$

are given, *and* that the principle of commutativity is assumed, so that if A and B commute with the X_i then A and B commute with each other. One then can interpret ∂_i as a standard derivative with $\partial_i(X_j) = \delta_{ij}$. Furthermore, one can verify

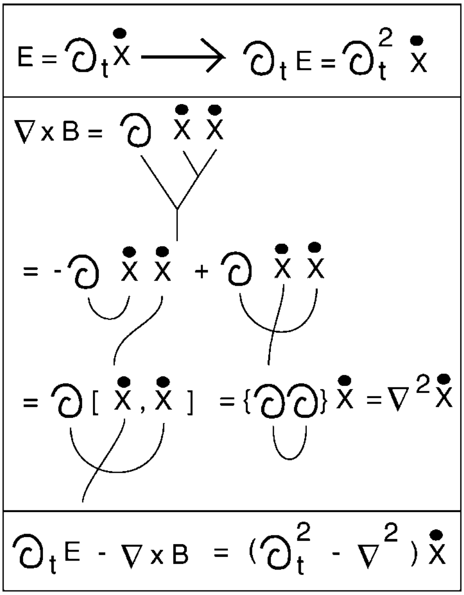


Fig 7. Curl of B.

that E_j and B_j both commute with the X_i. From this it follows that $\partial_t(E)$ and $\partial_t(B)$ have standard intepretations and that $B \times B = 0$. The above formulation of the Theorem adds the description of E as $\partial_t(\dot{X})$, a non-standard use of ∂_t in the original context of Feyman-Dyson, where ∂_t would only be defined for those A that commute with X_i. In the same vein, the last formula $\partial_t E - \nabla \times B = (\partial_t^2 - \nabla^2)\dot{X}$ gives a way to express the remaining Maxwell Equation in the Feynman-Dyson context.

Remark 3. Note the role played by the epsilon tensor ϵ_{ijk} throughout the construction of generalized electromagnetism in this section. The epsilon tensor is the structure constant for the Lie algebra of the rotation group $SO(3)$. If we replace the epsilon tensor by a structure constant f_{ijk} for a Lie algebra $\mathcal{G}$of dimension d such that the tensor is invariant under cyclic permutation ($f_{ijk} = f_{kij}$), then most of the work in this section will go over to that context. We would then have d operator/variables $X_1, \ldots X_d$ and a generalized cross product defined on vectors of length d by the equation

$$(A \times B)_k = f_{ijk} A_i B_j.$$

The Jacobi identity for the Lie algebra $\mathcal{G}$ implies that this cross product will satisfy

$$A \times (B \times C) = (A \times B) \times C + [B \times (A] \times C)$$

where

$$([B \times (A] \times C)_r = f_{klr} f_{ijk} A_i B_k C_j.$$

This extension of the Jacobi identity holds as well for the case of non-commutative cross product defined by the epsilon tensor. It is therefore of interest to explore the structure of generalized non-commutative electromagnetism over other Lie algebras (in the above sense). This will be the subject of another paper.

2.2. Discrete Thoughts

In the hypotheses of the Electromagnetic Theorem, we are free to take any non-commutative world, and the Electromagnetic Theorem will satisfied in that world. For example, we can take each X_i to be an arbitary time series of real or complex numbers, or bitstrings of zeroes and ones. The global time derivative is defined by

$$\dot{F} = J(F' - F) = [F, J],$$

where $FJ = JF'$. This is the non-commutative discrete context discussed in sections 1. We will write

$$\dot{F} = J\Delta(F)$$

where $\Delta(F)$ denotes the classical discrete derivative

$$\Delta(F) = F' - F.$$

With this interpretation X is a vector with three real or complex coordinates at each time, and

$$B = \dot{X} \times \dot{X} = J^2 \Delta(X') \times \Delta(X)$$

while

$$E = \ddot{X} - \dot{X} \times (\dot{X} \times \dot{X}) = J^2 \Delta^2(X) - J^3 \Delta(X'') \times (\Delta(X') \times \Delta(X)).$$

Note how the non-commutative vector cross products are composed through time shifts in this context of temporal sequences of scalars. The advantage of the generalization now becomes apparent. We can create very simple models of generalized electromagnetism with only the simplest of discrete materials. In the case of the model in terms of triples of time series, the generalized electromagnetic theory is a theory of measurements of the time series whose key quantities are

$$\Delta(X') \times \Delta(X)$$

and

$$\Delta(X'') \times (\Delta(X') \times \Delta(X)).$$

It is worth noting the forms of the basic derivations in this model. We have, assuming that F is a commuting scalar (or vector of scalars) and taking $\Delta_i = X'_i - X_i$,

$$\partial_i(F) = [F, \dot{X}_i] = [F, J\Delta_i] = FJ\Delta_i - J\Delta_i F = J(F'\Delta_i - \Delta_i F) = \dot{F}\Delta_i$$

and for the temporal derivative we have

$$\partial_t F = J[1 - J\Delta' \bullet \Delta]\Delta(F)$$

where $\Delta = (\Delta_1, \Delta_2, \Delta_3)$.

3. DIFFERENTIAL GEOMETRY AND GAUGE THEORY IN A NON-COMMUTATIVE WORLD

We take the dynamical law in the form

$$\frac{dX_i}{dt} = \dot{X}_i = P_i - A_i = \mathcal{G}_i.$$

This gives rise to new commutation relations

$$[X_i, \dot{X}_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - \frac{\partial A_j}{\partial P_i} = g_{ij}$$

where this equation defines g_{ij}, and

$$[\dot{X}_i, \dot{X}_j] = R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

We define the "covariant derivative"

$$\nabla_i F = [F, P_i - A_i] = \partial_i(F) - [F, A_i] = [F, \dot{X}_i],$$

while we can still write

$$\hat{\partial}_i F = [X_i, F].$$

It is natural to think that g_{ij} is analogous to a metric. This analogy is strongest if we *assume* that $[X_i, g_{jk}] = 0$. By assuming that the spatial coordinates commmute with the metric coefficients we have that

$$[\dot{X}_i, g_{jk}] + [X_i, \dot{g_{jk}}] = 0.$$

Hence

$$\nabla_i g_{jk} = \hat{\partial}_i \dot{g_{jk}}.$$

Here, we shall assume from now on that

$$[X_i, g_{jk}] = 0.$$

A stream of consequences then follows by differentiating both sides of the equation

$$g_{ij} = [X_i, \dot{X}_j].$$

We will detail these consequences in Section 4. For now, we show how the form of the Levi-Civita connection appears naturally.
In the following we shall use D as an abbreviation for d/dt.

The Levi-Civita connection

$$\Gamma_{ijk} = \left(\frac{1}{2}\right)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$$

associated with the g_{ij} comes up almost at once from the differentiation process described above. To see how this happens, view the following calculation where

$$\hat{\partial}_i \hat{\partial}_j F = [X_i, [X_j, F]].$$

We apply the operator $\hat{\partial}_i \hat{\partial}_j$ to the second time derivative of X_k.

Lemma 2. *Let* $\Gamma_{ijk} = (1/2)(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$. *Then*

$$\Gamma_{ijk} = (1/2)\hat{\partial}_i \hat{\partial}_j \ddot{X}_k.$$

Proof: Note that by the Leibniz rule

$$D([A, B]) = [\dot{A}, B] + [A, \dot{B}],$$

we have

$$\dot{g_{jk}} = [\dot{X}_j, \dot{X}_k] + [X_j, \ddot{X}_k].$$

Therefore

$$\begin{aligned}
\hat{\partial}_i \hat{\partial}_j \ddot{X}_k &= [X_i, [X_j, \ddot{X}_k]] \\
&= [X_i, \dot{g_{jk}} - [\dot{X}_j, \dot{X}_k]] \\
&= [X_i, \dot{g_{jk}}] - [X_i, [\dot{X}_j, \dot{X}_k]] \\
&= [X_i, \dot{g_{jk}}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]] \\
&= -[\dot{X}_i, g_{jk}] + [\dot{X}_k, [X_i, \dot{X}_j]] + [\dot{X}_j, [\dot{X}_k, X_i]]
\end{aligned}$$

$$
\begin{aligned}
&= \nabla_i g_{jk} - \nabla_k g_{ij} + \nabla_j g_{ik} \\
&= 2\Gamma_{kij}.
\end{aligned}
$$

This completes the proof. □

It is remarkable that the form of the Levi-Civita connection comes up directly from this non-commutative calculus without any apriori geometric interpretation.

The upshot of this derivation is that it confirms our interpretation of

$$g_{ij} = [X_i, \dot{X}_j] = [X_i, P_j] - [X_i, A_j] = \delta_{ij} - \frac{\partial A_j}{\partial P_i}$$

as an abstract form of metric (in the absence of any actual notion of distance in the non-commutative world). *This calls for a re-evaluation and reconstruction of differential geometry based on non-commutativity and the Jacobi identity.* This is differential geometry where the fundamental concept is no longer parallel translation, but rather a non-commutative version of a physical trajectory. This approach will be the subject of a separate paper.

At this stage we face the mystery of the appearance of the Levi-Civita connection. There is a way to see that the appearance of this connection is not an accident, but rather quite natural. We are thinking about the commutator $[X_i, \dot{X}_j] = g_{ij}$. It is worth noting that this equation is a consequence of the right choice of Hamiltonian. By this I mean, that in a given non-commutative world we choose an H in the algebra to represent the total (or discrete) time derivative so that $\dot{F} = [F, H]$ for any F. Suppose we have elements g_{ij} such that

$$[g_{ij}, X_k] = 0$$

and

$$g_{ij} = g_{ji}.$$

We choose

$$H = \frac{(g_{ij} P_i P_j + P_i P_j g_{ij})}{4}.$$

This is the non-commutative analog of the classical $H = (1/2)g_{ij} P_i P_j$. In the non-commutative case, there is no reason for the metric coefficients and the momenta P_i to commute since the metric coefficients are dependent on the positions X_j.

We now show that this choice of Hamiltonian implies that $[X_i, \dot{X}_j] = g_{ij}$. Once we see this consequence of the choice of the Hamiltonian, the appearance of the Levi-Civita connection is quite natural, since the classical case of a particle moving in generalized coordinates under Hamilton's equations implies geodesic motion in the Levi-Civita connection.

Lemma 3. *Let g_{ij} be given such that $[g_{ij}, X_k] = 0$ and $g_{ij} = g_{ji}$. Define*

$$H = \frac{(g_{ij} P_i P_j + P_i P_j g_{ij})}{4}$$

(where we sum over the repeated indices) and

$$\dot{F} = [F, H].$$

Then

$$[X_i, \dot{X}_j] = g_{ij}.$$

Proof: Consider

$$\begin{aligned}
[X_k, g_{ij} P_i P_j] &= g_{ij}[X_k, P_i P_j] \\
&= g_{ij}([X_k, P_i]P_j + P_i[X_k, P_j]) \\
&= g_{ij}(\delta_{ki} P_j + P_i \delta_{kj}) = g_{kj} P_j + g_{ik} P_i = 2g_{kj} P_j.
\end{aligned}$$

Then

$$\begin{aligned}
[X_r, \dot{X}_k] = [X_r, [X_k, H]] &= \left[X_r, \left[X_k, \frac{(g_{ij} P_i P_j + P_i P_j g_{ij})}{4}\right]\right] \\
&= \left[X_r, \left[X_k, \frac{(g_{ij} P_i P_j)}{4}\right]\right] + [X_r, [X_k, (P_i P_j g_{ij})/4]] \\
&= 2[X_r, 2g_{kj} P_j/4] = [X_r, g_{kj} P_j] = g_{kj}[X_r, P_j] = g_{kj}\delta_{rj} \\
&= g_{kr} = g_{rk}.
\end{aligned}$$

This completes the proof. □

It is natural to extend the present analysis to a discussion of general relativity. A joint paper on general relativity from this non-commutative standpoint is in preparation (joint work with Tony Deakin and Clive Kilmister).

4. CONSEQUENCES OF THE METRIC

In this section we shall follow the formalism of the metric commutator equation

$$[X_i, \dot{X}_j] = g_{ij}$$

very far in a semi-classical context. That is, we shall set up a non-commutative world, *and* we shall make assumptions about the non-commutativity that bring the operators into close analogy with variables in standard calculus. In particular we shall regard an element F of the Lie algebra to be a “function of the X_i” if F

commutes with the X_i, and we shall assume that if F and G commute with the X_i, then F and G commute with each other. We call this the *principle of commutativity*. With these background assumptions, it is possible to get a very sharp result about the behaviour of the theory. In particular, the results of this section sharpen the work in Tanimura (1992), where special orderings and averages of orderings of the operators were needed to obtain analogous results.

We assume that

$$[X_i, \dot{X}_j] = g_{ij}$$

$$[X_i, X_j] = 0$$

$$[X_i, g_{jk}] = 0$$

$$[g_{ij}, g_{kl}] = 0.$$

We assume that there exists a g^{ij} with

$$g^{ij} g_{jk} = \delta^i_k = g_{ij} g^{jk} = \delta^k_i.$$

We also assume that if

$$[A, X_i] = 0$$

and

$$[B, X_i] = 0$$

for all i, then

$$[A, B] = 0$$

for all expressions A and B in the algebra under consideration. To say that $[A, X_i] = 0$ is to say the analogue of the statement that A is a function only of the variables X_i and not a function of the $\dot{X}_j$. This is a stong assumption about the algebraic structure, and it will not be taken when we look at strictly discrete models. It is, however, exactly the assumption that brings the non-commutative algebra closest to the classical case of functions of positions and momenta.

The main result of this section will be a proof that

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t,$$

and that this decompositon of the acceleration is uniquely determined by the given framework. Since

$$F^{rs} = [\dot{X}^r, \dot{X}^s] = g^{ri} g^{sj} F_{ij},$$

we can regard this result as a description of the motion of the non-commutative particle influenced by a scalar field G_r, a qauge field F^{rs}, and geodesic motion with respect to the Levi-Civita connection corresponding to g_{ij}. Let us begin.

Note that, as before, we have that $g_{ij} = g_{ji}$ by taking the time derivative of the equation $[X_i, X_j] = 0$.

Note also that the Einstein summation convention (summing over repeated indices) is in effect when we write equations, unless otherwise specified.

As before, we define

$$\partial_i F = [F, \dot{X}_i]$$

and

$$\hat{\partial}_i F = [X_i, F].$$

We also make the definitions

$$\dot{X}^i = g^{ij} \dot{X}_j$$

and

$$\partial^i F = [F, \dot{X}^i].$$

Note that we do not assume the existence of a variable X^j whose time derivative is $\dot{X}^j$. Note that we have

$$\dot{X}_k = g_{ki} \dot{X}^i.$$

Note that it follows at once that

$$\hat{\partial}_i \dot{g}_{jk} = \partial_i g_{jk}$$

by differentiating the equation $[X_i, g_{jk}] = 0$.

We assume the following postulate about the time derivative of an element F with $[X_i, F] = 0$ for all k:

$$\dot{F} = (\partial_i F) \dot{X}^i.$$

This is in accord with the concept that F is a function of the variables X_i. Note that in one interpretation of this formalism, one of the variables X_i could be itself a time variable. In the next section, we shall return to three dimensions of space and one dimension of time, with a separate notation for the time variable. Here there is no restriction on the number of independent variables X_i.

We have the following Lemma.

Lemma 4.

1. $[X_i, \dot{X}^j] = \delta_i^j$.
2. $\partial_r(g^{ij}) g_{jk} + g^{ij} \partial_r(g_{jk}) = 0$.

3. $[X_r, \partial_i g_{jk}] = 0.$

Proof:

$$[X_i, \dot{X}^j] = [X_i, g^{jk}\dot{X}_k] = [X_i, g^{jk}]\dot{X}_k + g^{jk}[X_i, \dot{X}_k]$$
$$= g^{jk}[X_i, \dot{X}_k] = g^{jk}g_{ik} = g^{jk}g_{ki} = \delta_i^j.$$

The second part of the proposition is an application of the Leibniz rule:

$$0 = \partial_r\left(\delta_k^i\right) = \partial_r\left(g^{ij}g_{jk}\right) = \partial_r(g^{ij})g_{jk} + g^{ij}\partial_r(g_{jk}).$$

Finally,

$$[X_r, \partial_i g_{jk}] = [X_r, [g_{jk}, \dot{X}_i]] = -[\dot{X}_i, [X_r, g_{jk}]] - [g_{jk}, [\dot{X}_i, X_r]]$$
$$= -[\dot{X}_i, 0] + [g_{jk}, g_{ir}] = 0 + 0 = 0.$$

This completes the proof of the Lemma. □

It follows from this lemma that ∂^i can be regarded as $\partial/\partial X_i$.

We have seen that it is natural to consider the commutator of the velocities $R_{ij} = [\dot{X}_i, \dot{X}_j]$ as a field or curvature. For the present analysis, we would prefer the field to commute with all the variables X_k in order to identify it as a "function of the variables X_k." We shall find, by a computation, that R_{ij} does not so commute, but that a compensating factor arises naturally. The result is as follows.

Proposition 1. *Let* $F_{rs} = [\dot{X}_r, \dot{X}_s] + (\partial_r g_{ks} - \partial_s g_{kr})X^k$ *and* $F^{rs} = [\dot{X}^r, \dot{X}^s]$. *Then*

1. F_{rs} *and* F^{rs} *commute with the variables* X_k.
2. $F^{rs} = g^{ri}g^{sj}F_{ij}$.

Proof: of Proposition. We begin by computing the commutator of X_i and $R_{rs} = [\dot{X}_r, \dot{X}_s]$ by using the Jacobi identity.

$$[X_i, [\dot{X}_r, \dot{X}_s]] = -[\dot{X}_s, [X_i, \dot{X}_r]] - [\dot{X}_r, [\dot{X}_s, X_i]] = \partial_s g_{ir} - \partial_r g_{is}.$$

Note also that

$$[X_i, \partial_r g_{ks}] = [X_i, [g_{ks}, \dot{X}_r]] = -[\dot{X}_r, [X_i, g_{ks}]] - [g_{ks}, [\dot{X}_r, X_i]]$$
$$= -[\dot{X}_r, [X_i, g_{ks}]] + [g_{ks}, g_{ir}] = 0.$$

Hence

$$[X_i, (\partial_r g_{ks} - \partial_s g_{kr})X^k] = \partial_r g_{is} - \partial_s g_{ir}.$$

Therefore

$$[X_i, F_{rs}] = [X_i, [\dot{X}_r, \dot{X}_s] + (\partial_r g_{ks} - \partial_s g_{kr})X^k] = 0.$$

This, and an a similar computation that we leave for the reader, proves the first part of the proposition. We prove the second part by direct computation: Note the following identity:

$$[AB, CD] = [A, C]BD + A[B, C]D + C[A, D]B + CA[B, D].$$

Using this identity we find

$$\begin{aligned}
[\dot{X}^r, \dot{X}^s] &= [g^{ri}\dot{X}_i, g^{sj}\dot{X}_j] \\
&= [g^{ri}, g^{sj}]\dot{X}_i\dot{X}_j + g^{ri}[\dot{X}_i, g^{sj}]\dot{X}_j + g^{sj}[g^{ri}, \dot{X}_j]\dot{X}_i + g^{sj}g^{ri}[\dot{X}_i, \dot{X}_j] \\
&= -g^{ri}\partial_i(g^{sj})\dot{X}_j + g^{sj}\partial_j(g^{ri})\dot{X}_i + g^{sj}g^{ri}[\dot{X}_i, \dot{X}_j] \\
&= -g^{ri}\partial_i(g^{sj})g_{jl}\dot{X}^l + g^{sj}\partial_j(g^{ri})g_{il}\dot{X}^l + g^{sj}g^{ri}[\dot{X}_i, \dot{X}_j] \\
&= g^{ri}g^{sj}\partial_i(g_{jl})\dot{X}^l - g^{sj}g^{ri}\partial_j(g_{il})\dot{X}^l + g^{sj}g^{ri}[\dot{X}_i, \dot{X}_j] \\
&= g^{ri}g^{sj}(\partial_i(g_{jl})\dot{X}^l - \partial_j(g_{il})\dot{X}^l + [\dot{X}_i, \dot{X}_j]) \\
&= g^{ri}g^{sj}F_{ij}.
\end{aligned}$$

This completes the proof of the proposition. □

We now consider the full form of the acceleration terms $\ddot{X}_k$. We have already shown that

$$\hat{\partial}_i\hat{\partial}_j\ddot{X}_k = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}.$$

Letting

$$\Gamma_{kij} = (1/2)(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}),$$

we *define* G_r by the formula

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t.$$

Proposition 2. *Let Γ_{rst} and G_r be defined as above. Then both Γ_{rst} and G_r commute with the variables X_i.*

Proof: Since we know that $[X_l, \partial_i g_{jk}] = 0$, it follows at once that $[X_l, \Gamma_{rst}] = 0$. It remains to examine the commutator $[X_l, G_r]$. We have

$$\begin{aligned}
[X_l, G_r] &= [X_l, \ddot{X}_r - F_{rs}\dot{X}^s - \Gamma_{rst}\dot{X}^s\dot{X}^t] \\
&= [X_l, \ddot{X}_r] - [X_l, F_{rs}\dot{X}^s] - [X_l, \Gamma_{rst}\dot{X}^s\dot{X}^t] \\
&= [X_l, \ddot{X}_r] - F_{rs}[X_l, \dot{X}^s] - \Gamma_{rst}[X_l, \dot{X}^s\dot{X}^t]
\end{aligned}$$

(since F_{rs} and Γ_{rst} commute with X_l). Note that

$$[X_l, \dot{X}^s] = \delta_l^s$$

and that

$$[X_l, \dot{X}^s \dot{X}^t] = [X_l, \dot{X}^s]\dot{X}^t + \dot{X}^s[X_l, \dot{X}^t]$$

$$= \delta_l^s \dot{X}^t + \dot{X}^s \delta_l^t.$$

Thus

$$\begin{aligned}[X_l, G_r] &= [X_l, \ddot{X}_r] - F_{rs}\delta_l^s - \Gamma_{rst}(\delta_l^s \dot{X}^t + \dot{X}^s \delta_l^t) \\ &= [X_l, \ddot{X}_r] - F_{rl} - \Gamma_{rlt}\dot{X}^t - \Gamma_{rsl}\dot{X}^s.\end{aligned}$$

It is easy to see that $\Gamma_{rlt}\dot{X}^t = \Gamma_{rsl}\dot{X}^s$. Hence

$$[X_l, G_r] = [X_l, \ddot{X}_r] - F_{rl} - 2\Gamma_{rlt}\dot{X}^t.$$

On the other hand,

$$[X_l, \dot{X}_r] = g_{lr}.$$

Hence

$$[X_l, \ddot{X}_r] = \dot{g_{lr}} - [\dot{X}_l, \dot{X}_r].$$

Therefore

$$\begin{aligned}[X_l, G_r] &= \dot{g_{lr}} - [\dot{X}_l, \dot{X}_r] - F_{rl} - 2\Gamma_{rlt}\dot{X}^t \\ &= \dot{g_{lr}} - (\partial_r g_{kl} - \partial_l g_{kr})\dot{X}^k - 2\Gamma_{rlt}\dot{X}^t.\end{aligned}$$

(since $F_{rl} = [\dot{X}_r, \dot{X}_l] + (\partial_r g_{kl} - \partial_l g_{kr})\dot{X}^k$) Hence

$$\begin{aligned}[X_l, G_r] &= \dot{g_{lr}} - (\partial_r g_{tl} - \partial_l g_{tr})\dot{X}^t - (\partial_l g_{tr} + \partial_t g_{lr} - \partial_r g_{lt})\dot{X}^t \\ &= \dot{g_{lr}} - (\partial_t g_{lr})\dot{X}^t = 0.\end{aligned}$$

This completes the proof of the proposition. □

We now know that G_r, F_{rs} and Γ_{rst} commute with the variables X_k. As we now shall see, the formula

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t$$

allows us to extract these functions from $\ddot{X}_r$ by differentiating with respect to the dual variables. We already know that

$$\hat{\partial}_i \hat{\partial}_j \ddot{X}_k = 2\Gamma_{kij},$$

and now note that

$$\begin{aligned}\hat{\partial}_i(\ddot{X}_r) &= [X_i, \ddot{X}_r] = [X_i, G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t] \\ &= F_{rs}[X_i, \dot{X}^s] + \Gamma_{rst}[X_i, \dot{X}^s\dot{X}^t] \\ &= F_{ri} + 2\Gamma_{rit}\dot{X}^t.\end{aligned}$$

We see now that the decomposition

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t$$

of the acceleration is uniquely determined by these conditions. Since

$$F^{rs} = [\dot{X}^r, \dot{X}^s] = g^{ri}g^{sj}F_{ij},$$

we can regard this result as a description of the motion of the non-commutative particle influenced by a scalar field G_r, a qauge field F^{rs}, and geodesic motion with respect to the Levi-Civita connection corresponding to g_{ij}. The structural appearance of all of these physical aspects is a mathematical consequence of the choice of non-commutative framework.

Remark 4. It follows from the Jacobi identity that

$$F_{ij} = g_{ir}g_{js}F^{rs}$$

satisfies the equation

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0,$$

identifying F_{ij} as a non-commutative analog of a gauge field. G_i is a non-commutative analog of a scalar field. The derivation in this section generalizes the Feynman-Dyson derivation of non-commutative electromagnetism Dyson (1990) where $g_{ij} = \delta_{ij}$. In the next section we will say more about the Feynman-Dyson result. The results of this section sharpen considerably an approach of Tanimura (1992). In Tanimura's paper, normal ordering techniques are used to handle the algebra. In the derivation given above, we have used straight non-commutative algebra, just as in the original Feynman-Dyson derivation.

Remark 5. It is interesting to note that we can rewrite the equation

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t$$

as

$$\ddot{X}_r = G_r + [\dot{X}_r, \dot{X}_s]\dot{X}^s + \Gamma_{srt}\dot{X}^s\dot{X}^t.$$

(Just substitute the expression for F_{rs} and recollect the terms.) The reader may enjoy trying her hand at other ways to reorganize this data. It is important to note that in the first form of the equation, the basic terms G_r, F_{rs} and Γ_{rst} commute with the coordinates X_k. It is this decomposition into parts that commute with the coordinates that guides the structure of this formula in the non-commutative context.

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